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On geometry of hypersurfaces of a pseudoconformal space of Lorentzian signature

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Abstract

There are three types of hypersurfaces in a pseudoconformal space C_1^n of Lorentzian signature: spacelike, timelike, and lightlike. These three types of hypersurfaces are considered in parallel. Spacelike hypersurfaces are endowed with a proper conformal structure, and timelike hypersurfaces are endowed with a conformal structure of Lorentzian type. Geometry of these two types of hypersurfaces can be studied in a manner that is similar to that for hypersurfaces of a proper conformal space. Lightlike hypersurfaces are endowed with a degenerate conformal structure. This is the reason that their investigation has special features. It is proved that under the Darboux mapping such hypersurfaces are transferred into tangentially degenerate $(n - 1)$ -dimensional submanifolds of rank $n - 2$ located on the Darboux hyperquadric. The isotropic congruences of the space C_1^n that are closely connected with lightlike hypersurfaces and their Darboux mapping are also considered.

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1. Introduction

Submanifolds in a proper conformal space C^n were considered in numerous papers. Submanifolds in pseudo-Euclidean spaces, in particular, in the Minkowski space, were also investigated in great detail (see, e.g., [ON 83]). There are three types of submanifolds in a

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pseudo-Euclidean space: spacelike, timelike, and lightlike. These three types of submanifolds were also studied in pseudo-Riemannian spaces of different signatures. In the recently published book [DB 96] the geometry of lightlike hypersurfaces and lightlike submanifolds of higher codimension in semi-Riemannian (or in another terminology pseudo-Riemannian) spaces were investigated in detail.

However, the property of submanifolds to be spacelike, timelike or lightlike is invariant with respect to conformal transformations of the pseudo-Riemannian spaces in which they are embedded. This is the reason that it is appropriate to consider all three types of submanifolds (spacelike, timelike, and lightlike) in the framework of pseudoconformal structures.

In the present paper we study hypersurfaces in a pseudoconformal space C_1^n of Lorentzian signature. We show that the local theory of spacelike and timelike hypersurfaces in the space C_1^n can be developed along the same lines as the theory of hypersurfaces in a proper conformal space (Sections 4 and 5). The theory of lightlike (isotropic) hypersurfaces is quite different from the theory of hypersurfaces in a proper conformal space. We consider some aspects of the theory of lightlike hypersurfaces (Section 6) and isotropic congruences that are closely connected with lightlike hypersurfaces (Section 7). The use of pseudoconformal setting for studying of hypersurfaces allows us to apply the Darboux mapping, prove that under this mapping the image of a lightlike hypersurface is a tangentially degenerate submanifold in a projective space and describe singular points on a lightlike hypersurface and on an isotropic congruence of a pseudoconformal space.

Note that in [DB 96] the results on lightlike hypersurfaces in semi-Riemannian spaces are applied to electrodynamics and general relativity. But since many of these applications and the lightlike hypersurfaces themselves are conformally invariant, the results of the current paper can be used in similar and possibly other physical applications.

The isotropic congruences in pseudo-Riemannian spaces are of interest for general relativity. In particular, they are connected with construction of the Kerr metric describing black holes in the gravitational field (see [Ch 83]).

2. Preliminaries

It is well known that a geometric model of space–time in special relativity is the Minkowski space, i.e., a four-dimensional pseudo-Euclidean space R_1^4 of signature (3, 1) (see, e.g., [BEE 96]). The fundamental quadratic form of this space is reduced to the form

$$g = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 - (\omega^4)^2,$$

where $\omega^1, \omega^2, \omega^3$ are the space coordinates and ω^4 is the time coordinate in the tangent space T_x associated with a point x of the space R_1^4 . (The space T_x is the set of vectors of the space R_1^4 emanating from the point x .) The relatively invariant fundamental form g defines the Lorentzian metric in R_1^4 . The group of transformations of the space T_x preserving this metric is the pseudoorthogonal group $\mathbf{O}(3, 1)$ that is also called the Lorentz group.

The equation $g = 0$ determines in the space T_x the light cone C_x whose generators are light trajectories propagating from a source located at the point x . The group of

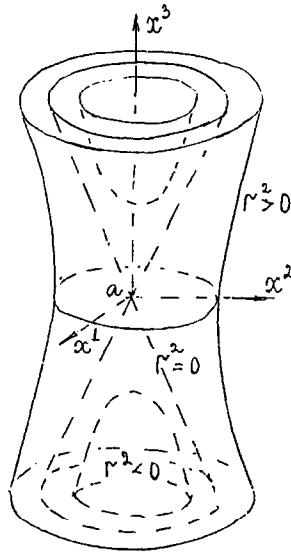


Fig. 1.

transformations of the space T_x leaving the cone C_x invariant is the group $G = \mathbf{O}(3, 1) \times \mathbf{H}$, where \mathbf{H} is the group of homotheties of T_x .

Many results of special relativity, especially results concerning the light propagation, are connected with the conformal structure of the space R_1^4 – the structure determined on R_1^4 by the relatively invariant fundamental form g . In fact, the equation $g = 0$ defines in T_x the light cone C_x with vertex at the point x , and the set of these cones is invariant under pseudoconformal transformations of the space R_1^4 . Besides the light cones, these transformations leave invariant the set of hyperspheres of the space R_1^4 defined in T_x by the equation $g = r^2$, where the number r^2 can be not only positive (as in the Euclidean space) but also negative or zero. For $r^2 < 0$, the equation $g = r^2$ determines hyperspheres of imaginary radius; for $r^2 > 0$, it determines hyperspheres of real radius; and for $r^2 = 0$, it determines hyperspheres of zero radius coinciding with the light cones (see Fig. 1 for $n = 3$).

Conformal transformations of the space R_1^4 form a group depending on 15 parameters. However, this group does not act bijectively in the space R_1^4 . To make this group to act bijectively on the set of points of the space R_1^4 , we should enlarge this set by ideal elements: a point at infinity $y = \infty$ and the light cone C_y with vertex at the point y . The enlarged space R_1^4 is denoted by C_1^4 and is called the *pseudoconformal space* of signature $(3, 1)$, $C_1^4 = R_1^4 \cup C_y$. After this enlargement, the noncompact space R_1^4 becomes the compact space C_1^4 . This is the reason that such an operation is called the *compactification* of the Minkowski space.

In what follows we will consider not only four-dimensional space C_1^4 but also n -dimensional spaces C_1^n of Lorentzian signature for $n \geq 3$. The fundamental form g defining a conformal structure of this space can be reduced to the form

$$g = (\omega^1)^2 + \dots + (\omega^{n-1})^2 - (\omega^n)^2. \tag{1}$$

The space C_1^n admits a one-to-one point mapping onto a hyperquadric Q_1^n of a projective space P^{n+1} . The equation of Q_1^n can be reduced to the form

$$(x^1)^2 + \dots + (x^{n-1})^2 - (x^n)^2 + (x^0)^2 - (x^{n+1})^2 = 0. \tag{2}$$

The projective coordinates x^0, \dots, x^{n+1} of points of the space P^{n+1} are called *polyspherical coordinates* of the elements (points and hyperspheres) of the space C_1^n (see [K1 26] or [AG 96]).

The quadratic form on the left-hand side of Eq. (2) determines the *scalar product* of elements of the space C_1^n . As usual, we denote this scalar product by (\cdot, \cdot) . The scalar square of a point of the space C_1^n is equal to 0, and it is negative for spacelike hyperspheres and positive for timelike hyperspheres. The vanishing of the scalar product of two hyperspheres means that the hyperspheres are orthogonal, and the vanishing of the scalar product of a point and a hypersphere means that the point belongs to the hypersphere.

The group of conformal transformations of the space C_1^n is isomorphic to the group of projective transformations of the space P^{n+1} sending the hyperquadric Q_1^n to itself. This group is denoted by $\mathbf{PO}(n + 2, 2)$ and is expressed as follows:

$$\mathbf{PO}(n + 2, 2) := \begin{cases} \mathbf{SO}(n + 2, 2) & \text{if } n \text{ is odd,} \\ \mathbf{O}(n + 2, 2)/\mathbf{Z}_2 & \text{if } n \text{ is even,} \end{cases}$$

where $\mathbf{O}(n + 2, 2)$ and $\mathbf{SO}(n + 2, 2)$ are the groups of pseudoorthogonal and special pseudoorthogonal transformations of the indicated signature, respectively, and \mathbf{Z}_2 is the cyclic group of second-order. In both cases this group depends on $\frac{1}{2}(n + 1)(n + 2)$ parameters.

The mapping $\varphi : C_1^n \rightarrow Q_1^n$ is called the *Darboux mapping*, and the hyperquadric Q_1^n is called the *Darboux hyperquadric*. Such a mapping was constructed first for the proper conformal three-dimensional space C^3 (see [K1 26]). Under the mapping φ to the isotropic cones C_x there correspond the *asymptotic cones* of the hyperquadric Q_1^n . This hyperquadric carries real rectilinear generators to which in the space C_1^n there correspond the lines of light propagation. The light cones in C_1^n are called also the *isotropic cones*, and the lines of light propagation are called the *isotropic lines* of the space C_1^n .

Further, we will apply the method of moving frames. In the space C_1^n we consider a family of conformal frames consisting of two points A_0 and A_{n+1} and n hyperspheres $A_r, r = 1, \dots, n$, passing through these points. The frame elements of such frames satisfy the following analytical conditions:

$$(A_0, A_0) = (A_{n+1}, A_{n+1}) = 0, \tag{3}$$

$$(A_0, A_0) = (A_{n+1}, A_{n+1}) = 0, \quad (A_0, A_r) = (A_{n+1}, A_r) = 0, \tag{4}$$

$$(A_r, A_s) = g_{rs}, \quad r, s = 1, \dots, n.$$

In addition, we normalize the points A_0 and A_{n+1} by the condition

$$(A_0, A_{n+1}) = -1. \tag{5}$$

Under the Darboux mapping to such frames there correspond point projective frames in the space P^{n+1} for which the points A_0 and A_{n+1} lie on the Darboux hyperquadric but do

not belong to any of its rectilinear generators, and the points A_r form a basis of the $(n - 1)$ -dimensional subspace that is polar-conjugate to the straight line A_0A_{n+1} with respect to the Darboux hyperquadric. With respect to this projective point frame the equation of Darboux hyperquadric takes the form

$$g_{rs}x^r x^s - 2x^0 x^{n+1} = 0, \tag{6}$$

where the quadratic form $g_{rs}x^r x^s$ is of signature $(n - 1, 1)$.

The equations of infinitesimal displacement of our conformal frame in the space C_1^n are

$$dA_\xi = \omega_\xi^\eta A_\eta, \quad \xi, \eta = 0, 1, \dots, n + 1, \tag{7}$$

where ω_ξ^η are differential 1-forms containing the parameters, on which the group $\mathbf{PO}(n + 2, 2)$ depends, and their differentials.

If we differentiate conditions (3)–(5) by means of Eq. (7), we obtain that the forms ω_ξ^η satisfy the following equations:

$$\omega_{n+1}^0 = \omega_0^{n+1} = 0, \quad \omega_0^0 + \omega_{n+1}^{n+1} = 0, \tag{8}$$

$$\omega_r^{n+1} - g_{rs}\omega_0^s = 0, \quad \omega_r^0 - g_{rs}\omega_{n+1}^s = 0, \tag{9}$$

$$dg_{rs} = g_{rt}\omega_s^t + g_{ts}\omega_r^t. \tag{10}$$

In addition, the forms ω_η^ξ satisfy the structure equations of the spaces C_1^n and P^{n+1} :

$$d\omega_\xi^\eta = \omega_\xi^\zeta \wedge \omega_\zeta^\eta \tag{11}$$

that are necessary and sufficient conditions for complete integrability of Eq (7).

3. Hypersurfaces in the space C_1^n

In the space C_1^n we consider a hypersurface V^{n-1} , i.e., a smooth, connected and simply connected submanifold of dimension $n - 1$. The conformal structure of the space C_1^n induces a conformal structure on the hypersurface V^{n-1} . The nature of this structure depends on the mutual location of tangent hyperplanes $T_x(V^{n-1}) = \tau_x, x \in V^{n-1}$, with respect to the isotropic cones C_x of the space C_1^n . Three “pure” cases of such location are possible:

- (a) At any point $x \in V^{n-1}$ the hyperplane τ_x and the isotropic cone C_x have only one common point x . Then the quadratic form $\tilde{g} = g|_{\tau_x}$ on the hypersurface V^{n-1} is positive definite, and on V^{n-1} a proper conformal structure is induced. A hypersurface V^{n-1} of this type is called *spacelike*.
- (b) At any point $x \in V^{n-1}$ the hyperplane τ_x intersects the isotropic cone C_x along a real cone \tilde{C}_x of dimension $n - 2$. Then the quadratic form \tilde{g} on V^{n-1} has signature $(n - 2, 1)$, and on V^{n-1} a conformal structure of the same signature is induced. A hypersurface V^{n-1} of this type is called *timelike*.
- (c) At any point $x \in V^{n-1}$ the hyperplane τ_x is tangent to the isotropic cone C_x . Then the quadratic form \tilde{g} on V^{n-1} has signature $(n - 2, 0)$. A hypersurface V^{n-1} of this type is called *lightlike* or *isotropic*.

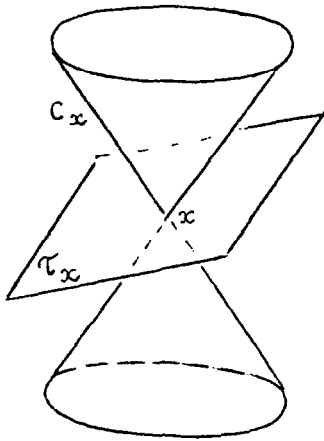


Fig. 2.

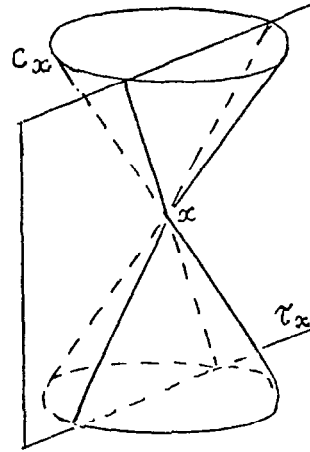


Fig. 3.

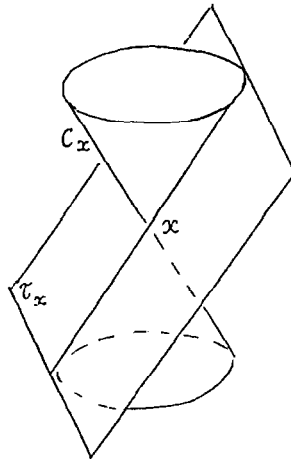


Fig. 4.

For the dimension $n = 3$ these three cases are represented on Figs. 2, 3, and 4.

The terminology (spacelike, timelike and lightlike) is related to that of general relativity. As was noted in Section 1, space–time in special relativity is a Minkowski space. In general relativity it is a pseudo-Riemannian space.

In both cases its metric has the signature $(3, 1)$ (or $(1, 3)$ – this depends on the method of presentation). In general relativity the isotropic cone C_x plays the role of the light cone. This cone divides the tangent space $T_x(C_1^n)$ (or space $T_x(C_{n-1}^n)$) into two domains – internal and external. Directions belonging to the first domain are called timelike, and directions belonging to the second domain are called spacelike (see Fig. 5). The tangent hyperplane $T_x(V^{n-1})$ to a spacelike hypersurface contains only directions located outside of the cone

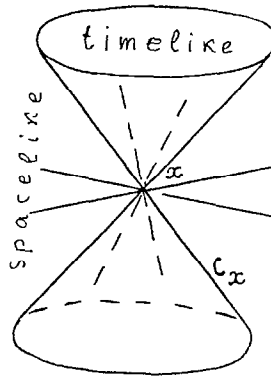


Fig. 5.

C_x , namely spacelike directions. For a timelike hypersurface V^{n-1} the tangent hyperplane $T_x(V^{n-1})$ contains both spacelike and timelike directions.

Note that hyperspheres of real radius, defined in the space T_x by the equation $g = a$ where $a > 0$, are spacelike hypersurfaces without singularities. If $a < 0$, then the equation $g = a$ defines timelike hypersurfaces also not having singularities. Finally, if $a = 0$, then the equation $g = a$ defines a hypersphere of zero radius, i.e., a lightlike hypersurface with the only singular point x . For $n = 3$, such hypersurfaces are presented in Fig. 1.

Note also that although under conformal transformations hyperspheres are transferred into hyperspheres, the radii of these hyperspheres are not invariant. However, under conformal transformations the nature of hyperspheres (i.e., their property to be spacelike, or timelike or lightlike) is invariant.

Besides “pure” hypersurfaces indicated above, there are hypersurfaces having points of two or of all three types indicated above. However, we will not consider such hypersurfaces in the present paper.

4. Geometry of spacelike hypersurfaces

We will study now the geometry of spacelike hypersurfaces V^{n-1} of the pseudoconformal space C_1^n in more detail.

With each point x of the hypersurface V^{n-1} , we associate a family of conformal frames in such a way that $A_0 = x$, the hypersphere A_n is tangent to V^{n-1} at the point x , and the hyperspheres A_i , $i = 1, \dots, n - 1$, are orthogonal to V^{n-1} at this point. Hence, the hypersphere A_n is spacelike, and hyperspheres A_i are timelike.

After such a specialization of moving frames Eqs. (3) and (5) will not be changed as well as the first two groups of Eq. (4) while the third group of Eq. (4) takes the form

$$(A_i, A_n) = 0, \quad (A_i, A_j) = g_{ij}, \quad (A_n, A_n) = -1, \tag{12}$$

where (g_{ij}) is a nondegenerate symmetric matrix of coefficients of a positive definite quadratic form \tilde{g} . Note that the last equation in (12) is obtained by means of an additional normalization of the hypersphere A_n : this normalization is possible, since A_n is a spacelike hypersphere. With respect to this frame the equation of the Darboux hyperquadric takes the form

$$g_{ij}x^i x^j - (x^n)^2 - 2x^0 x^{n+1} = 0. \tag{13}$$

Since the hypersphere A_n is tangent to the hypersurface V^{n-1} at the point A_0 , we have $(dA_0, A_n) = 0$. By the first equation of (7), this implies

$$\omega_0^n = 0 \tag{14}$$

and

$$dA_0 = \omega_0^0 A_0 + \omega^i A_i, \tag{15}$$

where $\omega^i = \omega_0^i$. From (15) it follows that the forms ω^i are linearly independent.

The family of frames described above is the bundle $\mathcal{R}^1(V^{n-1})$ of first-order frames associated with the hypersurface V^{n-1} . A base of this frame bundle is the hypersurface V^{n-1} , and its fiber is the collection of frames with a fixed point $x = A_0$. The forms ω^i are *base forms* of $\mathcal{R}^1(V^{n-1})$, and the forms $\omega_0^0, \omega_i^0, \omega_i^j$ and ω_n^0 are its *fiber forms*.

The quadratic form g , defining the conformal structure in the space C_1^n at the point x , is expressed now as

$$g = g_{ij}\omega^i \omega^j - (\omega^n)^2,$$

and its restriction to the hypersurface V^{n-1} becomes

$$\tilde{g} = g_{ij}\omega^i \omega^j. \tag{16}$$

The form \tilde{g} is positive definite and defines a proper conformal structure on the hypersurface V^{n-1} . The coefficients g_{ij} of this quadratic form generate a $(0, 2)$ -tensor. This tensor is associated with a first-order neighborhood of the hypersurface V^{n-1} , since by (15) we have

$$\tilde{g} = (dA_0, dA_0).$$

We will not write here all equations which the forms ω_ξ^n satisfy on the hypersurface V^{n-1} and equations obtained as differential prolongations of Eq. (14). They differ unessentially from similar equations in the theory of hypersurfaces in a proper conformal space C^n . The latter theory was considered in [A 52,SS 80] (see also [AG 96]). This is the reason that we are not going to consider in detail this construction as well as other topics of the theory of spacelike hypersurfaces which are known for hypersurfaces of the proper conformal space C^n .

5. Geometry of timelike hypersurfaces

Suppose now that a hypersurface $V^{n-1} \subset C_1^n$ is timelike. Then at any point $x \in V^{n-1}$ its tangent hyperplane $T_x(V^{n-1}) = \tau_x$ is located with respect to the light cone C_x as indicated

in Fig. 3. The tangent hyperspheres to the hypersurface V^{n-1} are timelike. Thus, they can be normalized by the condition

$$(A_n, A_n) = 1.$$

A timelike hypersurface V^{n-1} is also determined by Eq. (14).

The fundamental form g defining the conformal structure of the space C_1^n is expressed now as

$$g = g_{ij}\omega^i\omega^j + (\omega^n)^2,$$

and its restriction \tilde{g} to V^{n-1} becomes

$$\tilde{g} = g_{ij}\omega^i\omega^j.$$

However, unlike for spacelike hypersurfaces, for the timelike hypersurfaces the form \tilde{g} is of signature $(n-2, 1)$. Thus, a timelike hypersurface V^{n-1} possesses a pseudoconformal structure of Lorentzian signature.

However, again the system of equations associated with a timelike hypersurface V^{n-1} differs unessentially from similar equations in the theory of hypersurfaces in a proper conformal space C^n . Thus, we will not go into details of investigation of timelike hypersurfaces.

Note only that since the isotropic cone \tilde{C}_x of a timelike hypersurface is real, its mutual location with the cone $a_{ij}\omega^i\omega^j = 0$, determined by the second fundamental tensor a_{ij} of V^{n-1} and connected with a second-order neighborhood of a point $x \in V^{n-1}$, can be more diverse than for a hypersurface of the space C^n or for a spacelike hypersurface of the space C_1^n . It would be interesting to construct a classification of timelike hypersurfaces based on the location of these two cones.

6. Geometry of lightlike hypersurfaces

Next we consider lightlike hypersurfaces of the space C_1^n . For such hypersurfaces the quadratic form \tilde{g} is of signature $(n-2, 0)$, and they carry degenerate conformal structures.

Our considerations will be simpler if we consider the Darboux mapping of a lightlike hypersurface $V^{n-1} \subset C_1^n$ and all geometric objects associated with this hypersurface. The hypersurface V^{n-1} will be mapped onto a submanifold U^{n-1} of dimension $n-1$ belonging to the Darboux hyperquadric that is determined in the space P^{n+1} by Eq. (6).

As usual we locate the vertex A_0 of the moving frame at the varying point $x \in U^{n-1}$ and the vertices A_1, \dots, A_{n-1} in the tangent $(n-1)$ -plane $T_x(U^{n-1})$. Then the equation

$$\omega_0^n = 0 \tag{17}$$

holds.

But since the hypersurface V^{n-1} is lightlike, the tangent $(n-1)$ -plane $T_x(U^{n-1})$ is tangent to the asymptotic cone C_x of the Darboux hyperquadric. The latter cone corresponds to the

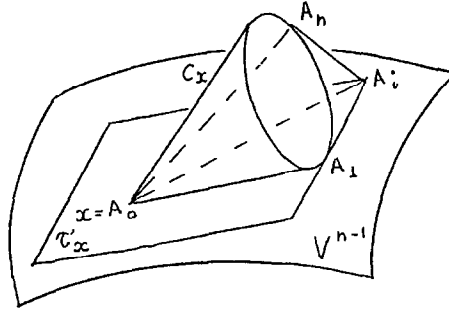


Fig. 6.

isotropic cone C_x of the space C_1^n . We place the vertex A_1 on the rectilinear generator along which the cone C_x is tangent to the subspace $T_x(U^{n-1})$. We also place the vertex A_n on the cone C_x but outside of this tangent subspace $T_x(U^{n-1})$ (see Fig. 6). Then in addition to Eqs. (3)–(5) which the elements of a moving frame satisfy, we have also the following relations:

$$(A_1, A_1) = (A_n, A_n) = (A_0, A_1) = (A_0, A_n) = 0. \tag{18}$$

Moreover, we normalize the points A_1 and A_n by the condition

$$(A_1, A_n) = -1. \tag{19}$$

By virtue of this, the matrix of the scalar products of the elements of the moving frame takes the form

$$(A_\xi, A_\eta) = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & g_{ij} & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{20}$$

where $\xi, \eta = 0, 1, \dots, n + 1; i, j = 2, \dots, n - 1$. As a result, the equation of the Darboux hyperquadric takes the form

$$g_{ij}x^i x^j - 2x^1 x^n - 2x^0 x^{n+1} = 0, \tag{21}$$

where $g_{ij}x^i x^j$ is a positive definite quadratic form.

It follows that the $(n - 3)$ -dimensional subspace, determined in the space P^{n+1} by the points $A_i, i = 2, \dots, n - 1$, does not have real common points with the Darboux hyperquadric, and the subspace, which is polar-conjugate to the above subspace with respect to this hyperquadric and is determined by the points A_0, A_1, A_n and A_{n+1} , intersects this hyperquadric in the following ruled surface of second order:

$$x^1 x^n + x^0 x^{n+1} = 0.$$

The above four points are located on this ruled surface as indicated in Fig. 7.

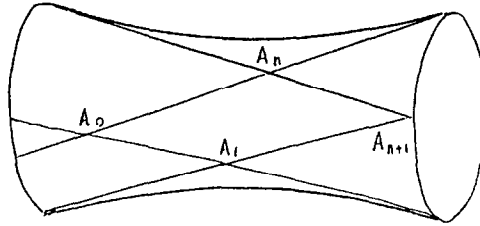


Fig. 7.

The equation of the asymptotic cone C_x at the point $x = A_0$ of the Darboux hyperquadric has the form

$$g = g_{ij}\omega^i\omega^j - 2\omega^1\omega^n = 0. \tag{22}$$

Since the equation of the hypersurface V^{n-1} has the form (17), the equation of the cone \tilde{C}_x of the submanifold U^{n-1} as well as of that of the hypersurface V^{n-1} has the form

$$\tilde{g} = g_{ij}\omega^i\omega^j = 0, \quad i, j = 2, \dots, n - 1. \tag{23}$$

This implies that at the point x this light cone has a single rectilinear generator A_0A_1 along which the subspace $T_x(U^{n-1})$ is tangent to the asymptotic cone C_x .

Next we write the equations of infinitesimal displacement of the moving frame associated with the point $x \in U^{n-1} \subset Q_1^n \subset P^{n+1}$ in the form (7) where the 1-forms ω_ξ^n satisfy the Eqs. (8)–(10) and also the equations obtained by differentiation of Eqs. (18) and (19):

$$\omega_1^n = 0, \quad \omega_n^1 = 0, \quad \omega_0^n + \omega_1^{n+1} = 0, \quad \omega_0^1 + \omega_n^{n+1} = 0, \quad \omega_1^1 + \omega_n^n = 0. \tag{24}$$

If we also differentiate the equation $g_{1i} = 0$, we find that

$$\omega_i^n = g_{ij}\omega_1^j, \quad i, j = 2, \dots, n - 1.$$

Since the tensor g_{ij} is nondegenerate, it follows from the last equation that

$$\omega_1^i = g^{ij}\omega_j^n. \tag{25}$$

Next, taking exterior derivatives of Eq. (17) and taking into account the first equation of (24), we obtain

$$\omega_i^n \wedge \omega_0^i = 0, \quad i = 2, \dots, n - 1. \tag{26}$$

Applying Cartan’s lemma to Eq. (26), we find that

$$\omega_i^n = \lambda_{ij}\omega_0^j, \quad i, j = 2, \dots, n - 1,$$

where $\lambda_{ij} = \lambda_{ji}$. Taking into account Eq. (25), we find

$$\omega_1^i = g^{ik}\lambda_{kj}\omega_0^j = \lambda_j^i\omega_0^j, \tag{27}$$

where $\lambda_j^i = g^{ik}\lambda_{kj}$ is a symmetric nondegenerate affinor.

We consider now the differentials of the points A_0 and A_1 . By (17) and (7), we obtain

$$dA_0 = \omega_0^0 A_0 + \omega_0^1 A_1 + \omega_0^i A_i, \quad dA_1 = \omega_1^0 A_0 + \omega_1^1 A_1 + \omega_1^i A_i. \quad (28)$$

From Eqs. (27) and (28) it follows that if $\omega_0^i = 0$, then the point A_0 moves along the lightlike straight line $A_0 A_1$ belonging to the cone C_x and describes the entire line $A_0 A_1$. This means that the submanifold U^{n-1} is a ruled submanifold. Moreover, the 1-form ω_0^1 defines the displacement of the point A_0 along the straight line $A_0 A_1$.

Next, Eqs. (28) show that at any point of the straight line $A_0 A_1$, the tangent $(n - 1)$ -dimensional subspace is fixed and coincides with the subspace $T_x(U^{n-1}) = A_0 \wedge A_1 \wedge A_2 \wedge \dots \wedge A_{n-1}$. Thus, the submanifold U^{n-1} is tangentially degenerate of rank $n - 2$ (see [AG 93, Ch. 4]), since the tangent subspace $T_x(U^{n-1})$ depends precisely on $n - 2$ parameters.

Let $X = A_1 + xA_0$ be an arbitrary point of the rectilinear generator $A_0 A_1$ of the submanifold U^{n-1} . Its differential is determined by the formula

$$dX \equiv (\omega_1^i + x\omega_0^i)A_i \pmod{A_0, A_1}.$$

Since, by (27),

$$\omega_1^i + x\omega_0^i = (\lambda_j^i + x\delta_j^i)\omega_0^j,$$

there are singular points on the straight line $A_0 A_1$, and their coordinates are determined by the equation

$$\det(\lambda_j^i + x\delta_j^i) = 0. \quad (29)$$

Since the tensor λ_j^i is symmetric, this equation has $n - 2$ real roots if we count each root as many times as its multiplicity.

Thus, we have proved the following result.

Theorem 1. *Under the Darboux mapping, to a lightlike hypersurface V^{n-1} of the pseudo-conformal space C_1^n there corresponds a ruled tangentially degenerate submanifold U^{n-1} of rank $n - 2$ whose rectilinear generator carries $n - 2$ real singular points if each of them is counted as many times as its multiplicity. These points are the images of singular points of the lightlike hypersurface V^{n-1} .*

The loci of singular points on lightlike hypersurfaces V^{n-1} are submanifolds whose dimension is less than $n - 1$. These submanifolds are called *focal submanifolds*. The dimension of focal submanifolds depends on the multiplicity of their elements – singular points.

If x_1 is a simple root of Eq. (29), then to this root there corresponds a family of torses (developable surfaces) on the submanifold U^{n-1} which are defined by the system of equations

$$\omega_1^i + x_1\omega_0^i = 0. \quad (30)$$

From the well-known theorem of linear algebra on orthogonality of eigendirections of a symmetric linear operator, it follows that to distinct roots of Eq. (29) there correspond two

mutual orthogonal families of torses on U^{n-1} . It is not difficult also to describe submanifolds on U^{n-1} corresponding to multiple roots of Eq. (29).

Note that since in general relativity, to lightlike straight lines of the space C_1^4 there correspond lines of propagation of light, then to singular points on lightlike hypersurfaces there correspond sources of light or points of its absorption, and their focal submanifolds are lighting surfaces or surfaces of light absorption. The further study of lightlike hypersurfaces in the space C_1^4 can be of interest for general relativity.

Note that the theory of lightlike hypersurfaces in semi-Riemannian spaces was studied in detail in [DB 96] and that some problems of the global theory of such hypersurfaces were considered by Kossowski (see, e.g., [Ko 89]).

7. Isotropic congruences

The notion of isotropic congruences of the space C_1^n is closely connected with the theory of isotropic hypersurfaces. An *isotropic congruence* is an $(n - 1)$ parameter family of isotropic straight lines such that through a generic point lying in a sufficiently small neighborhood of a straight line of the family there passes a unique straight line of the family.

To study the isotropic congruences we will apply again the Darboux mapping of the space C_1^n . Consider the set U of rectilinear generators of the Darboux hyperquadric Q_1^n . With any rectilinear generator of Q_1^n we associate a family of frames described in Section 6. Then with respect to any such frame the Darboux hyperquadric is defined by Eq. (21), and the components of infinitesimal displacements of these frames satisfy Eqs. (8)–(10) and (24).

Consider the rectilinear generator A_0A_1 . We have

$$dA_0 = \omega_0^0 A_0 + \omega_0^1 A_1 + \omega_0^i A_i + \omega_0^n A_n \tag{31}$$

and

$$dA_1 = \omega_1^0 A_0 + \omega_1^1 A_1 + \omega_1^i A_i - \omega_1^n A_{n+1}, \tag{32}$$

where $i = 2, \dots, n - 1$. On the hyperquadric Q_1^n the forms ω_0^i, ω_1^i , and ω_0^n are linearly independent, and their number is equal to $2n - 3$. Thus *the set U is a differentiable manifold of dimension $2n - 3$.*

The congruence of isotropic straight lines is an $(n - 1)$ -dimensional submanifold S of the manifold U . In general, this submanifold can be given on U by the following system of $n - 2$ equations:

$$\omega_1^i = \lambda_j^i \omega_0^j + \lambda^i \omega_0^n, \quad i, j = 1, 2, \dots, n - 1. \tag{33}$$

The forms ω_0^i and ω_0^n are basis forms of the congruence S .

On the congruence S Eq. (32) takes the form

$$dA_1 = \omega_1^0 A_0 + \omega_1^1 A_1 + \lambda_j^i \omega_0^j A_i - \omega_0^n (A_{n+1} - \lambda^i A_i). \tag{34}$$

In the projective space P^{n+1} the straight lines of the congruence in question describe a hypersurface that we will also denote by S . As a point set, the hypersurface S coincides with an open domain of the hyperquadric Q_1^n .

Let us study properties of the hypersurface S . Eqs. (31) and (34) imply that the linear span of a first-order neighborhood of the generator A_0A_1 coincides with the entire space P^{n+1} . Next all tangent hyperplanes $T_x(S)$ at the points of its generator A_0A_1 have the common subspace $A_0 \wedge A_1 \wedge A_2 \wedge \dots \wedge A_{n-1}$.

Consider singular points of the hypersurface S . Its point $X = A_1 + xA_0$ is singular if at this point the dimension of the tangent subspace $T_x(S)$ is less than n . By (31) and (34) we have

$$dX \equiv (\lambda_j^i + x\delta_j^i)\omega_0^j A_i + \omega_0^n (xA_n - A_{n+1} + \lambda^i A_i) \pmod{A_0, A_1}.$$

Thus, the dimension of the tangent subspace $T_x(S)$ is less than n if and only if

$$\det(\lambda_j^i + x\delta_j^i) = 0. \quad (35)$$

Eq. (35) determines singular points on the hypersurface S . Eq. (35) differs from Eq. (29), determining singular points on a lightlike hypersurface of the space C_1^n , only by the fact that the affiner λ_j^i was symmetric in (29) and is not symmetric in (35).

Now suppose that the equation $\omega_0^n = 0$ is completely integrable. Then the hypersurface S is stratified into a one-parameter family of $(n-1)$ -dimensional submanifolds to which in the space C_1^n there correspond lightlike hypersurfaces V^{n-1} .

The condition of complete integrability of the equation $\omega_0^n = 0$ has the form $d\omega_0^n \wedge \omega_0^n = 0$. By structure equations (11), this implies that in Eq. (33) the affiner λ_j^i is symmetric. As a result, all singular points of a rectilinear generator of the ruled hypersurface S are real.

Thus the following theorem is valid.

Theorem 2. *Any rectilinear generator A_0A_1 of the isotropic congruence S carries $n-2$ singular points if each of them is counted as many times as its multiplicity, and some of these singular points or all of them can be complex. These singular points are the Darboux images of the singular points of the congruence of the space C_1^n . If equation $\omega_0^n = 0$ is completely integrable on S , then it determines a stratification of the congruence S into lightlike hypersurfaces, and all these singular points are real.*

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